FLOW OF A VISCOUS INCOMPRESSIBLE FLUID IN A THIN LAYER CONFINED BY A RIGID FREE SURFACE

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In various applications problems often arise on calculating flow characteristics of an incompressible fluid in thin layers confined by rigid surfaces, one of which is fixed and the other free and under the action of a given system of forces. This kind of problem occurs in the hydrodynamic theory of lubrication [1-3], contact hydrodynamics [4], and some other applications. In this case it is usually assumed [1-3] that the kinematics of the rigid free surface is known ahead of time (the law of motion of the free surface is given, or its equilibrium state is considered). This approach does not always allow one to take into account the interaction of the fluid flow and the dynamics of the rigid free surface confining this flow region.

In the present study we consider an approach to constructing an approximate solution of three-dimensional nonstationary problems of the type indicated. In this case we consider simultaneously the problem of fluid flow in a thin layer with the dynamics problem of a rigid free surface confining this layer. The possibilities of this approach are illustrated on an example of two problems.

1. Consider laminar nonstationary flow of a viscous incompressible fluid in a region confined by two rigid surfaces, the characteristic distance between which being small in comparison with their longitudinal sizes. It is assumed that one of the surfaces, being a plane, is immobile, and the other is free and under the action of a given system of external forces.

Introduce a rectangular coordinate system, whose x and y axes are located in the fixed plane, with the z axis perpendicular to it. Neglecting then mass forces, and taking into account the assumption made above concerning the relation between longitudinal and transverse sizes of the region considered, the fluid flow is described by the system of equations

$$\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u_x}{\partial z^2},$$
$$\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\frac{\partial^2 u_y}{\partial z^2},\quad \frac{\partial p}{\partial z}=0; \tag{1.1}$$

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0 \tag{1.2}$$

with boundary conditions for the fluid velocity in the transverse coordinate

$$z=0,\ \mathbf{u}=\mathbf{V}; \tag{1.3}$$
$$z=h,\ \mathbf{u}=\mathbf{W} \tag{1.4}$$

(**V**, **W** are the values of the velocity vector **u** on the fixed and free surfaces). It is assumed here that the free surface is described in the selected reference system by a known function of its arguments of the form

$$h=h(x,\ y,\ q_1,\ q_2,\ \ldots), \tag{1.5}$$

where q_s(s = 1, 2, ..., S) are generalized coordinates of the rigid free surface, and S is the number of degrees of freedom of this surface. The vector **W**, being the velocity of the point of the free surface, whose coordinates equal {x; y}, is also given by a known function of its arguments

$$\mathbf{W}=\mathbf{W}(x,\ y,\ q_1,\ q_2,\ \ldots,\ \dot{q}_1,\ \dot{q}_2,\ \ldots) \tag{1.6}$$

($\dot{q}_s$(s = 1, 2, ..., S) are generalized velocities of the free surface).

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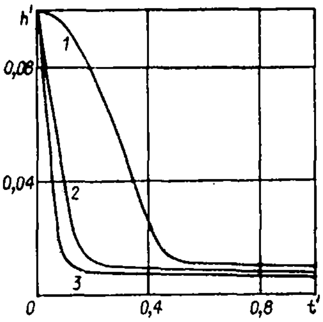

Fig. 1

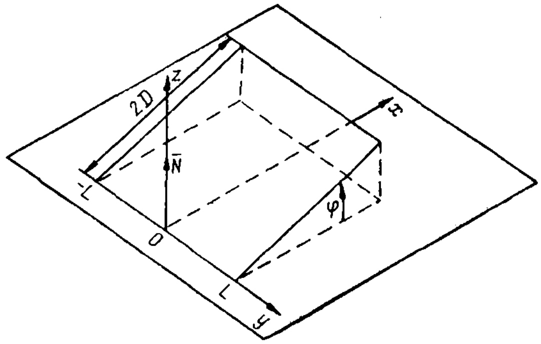

Fig. 2

Despite the fact that the second surface, confined by the flow region, is assumed immobile, in the general case $\mathbf{V} \not\equiv 0$, since on this surface, for example, one can have fluid supply (or suction) [5]. As to initial and boundary conditions in the longitudinal coordinates for (1.1), (1.2), we only note that their shape is determined separately for each specific problem.

The presence in the boundary conditions (1.4), with account of (1.5), (1.6), of generalized coordinates and free surface velocities, being unknown functions of time, causes the necessity of supplementing (1.1), (1.2) with a corresponding system of equations, describing the free surface dynamics. The role of such a closed system of equations is, obviously, played by the Lagrange equation of the second kind

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} = Q_s \quad (s = 1, 2, \ldots, S). \tag{1.7}$$

Here T is the kinetic energy of the free surface, and Q_s are the generalized forces of the system. We note that in determining Q_s it is necessary to take into account not only the external given forces, but also the stresses in the fluid, generated at the boundary with a free rigid surface.

The solution of the problem for the longitudinal velocity components will be sought in the form

$$u_x = \sum_{j=0}^{\infty} z^j u_{xj}, \quad u_y = \sum_{j=0}^{\infty} z^j u_{yj}, \tag{1.8}$$

where u_{xj}, u_{yj} are the yet unknown functions of time and of longitudinal coordinates. With account of (1.8) we obtain from (1.2), invoking the boundary condition (1.3) for u_z,

$$u_z = V_z - \sum_{j=0}^{\infty} \frac{z^{j+1}}{(j+1)} \left(\frac{\partial u_{xj}}{\partial x} + \frac{\partial u_{yj}}{\partial y}\right). \tag{1.9}$$

Substituting (1.8), (1.9) into (1.1), carrying out transformations, and reindexing, we find

$$\frac{\partial u_{\xi j}}{\partial t} + (j+1) V_z u_{\xi,j+1} - (j+1)(j+2)\frac{\mu}{\rho} u_{\xi,j+2} + \frac{\delta_{j0}}{\rho}\frac{\partial p}{\partial \xi} +$$
$$+ \sum_{i=0}^{j} \left\{ u_{xi} \frac{\partial u_{\xi,j-i}}{\partial x} + u_{yi} \frac{\partial u_{\xi,j-i}}{\partial y} - \frac{i u_{\xi i}}{(j-i+1)} \left(\frac{\partial u_{x,j-i}}{\partial x} + \frac{\partial u_{y,j-i}}{\partial y} \right) \right\} = 0$$
$$(j = 0, 1, \ldots). \tag{1.10}$$

Here and in the following for brevity ξ acquires the values x and y, respectively, for the first and second equation of (1.1), and δ_{j0} is the Kronecker symbol.

The special feature of the system obtained (1.10) is that in constructing the approximate solution, retaining in it for each ξ the first J equations, we always reach a subsystem of 2J equations containing 2J + 5 unknown functions:

$$p,\ u_{xj},\ u_{yj},\ (j = 0, 1, \ldots, J+1). \tag{1.11}$$

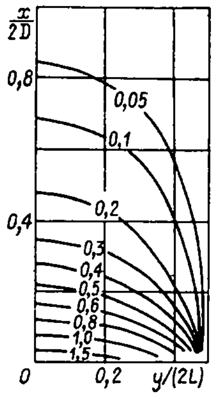

Fig. 3

The first four deficient equations follow from the boundary conditions (1.3), (1.4) for the longitudinal velocity components, and, with account of (1.8), are

$$u_{x0} = V_x, \quad u_{y0} = V_y, \quad W_x = \sum_{j=0}^{J+1} h^j u_{xj}, \quad W_y = \sum_{j=0}^{J+1} h^j u_{yj}. \tag{1.12}$$

The last deficient equation can be obtained from the boundary condition (1.4) for u_z, and with account of (1.9) is represented in the form

$$W_z = V_z - \sum_{j=0}^{J+1} \frac{h^{j+1}}{(j+1)} \left(\frac{\partial u_{xj}}{\partial x} + \frac{\partial u_{yj}}{\partial y} \right). \tag{1.13}$$

Thus, the final construction of the approximate solution of these problems reduces to a simultaneous solution of a system of 2J + S + 5 equations, consisting of (1.7), (1.12), (1.13), and the 2J first equations (J equations for each of the two ξ) of system (1.10). The solution of this system must be carried out both with account of initial conditions for the generalized coordinates and velocities of the free surface, and with initial and boundary conditions in the longitudinal coordinates for the functions (1.11). The latter conditions must be obtained from the corresponding original initial and boundary conditions for u_x and u_y with account of (1.8). We provide an approximate solution of several problems of the type indicated.

2. Consider fluid flow in the region between two parallel plates of length 2L and width 2D, one of which is immobile and the second is free and under the action of a constant force G normal to this plate. For this statement of the problem the mobile plate is a system with one degree of freedom, for whose generalized coordinate we choose the distance h to the fixed plate. The solution of the problem of a compressed fluid film between parallel rigid plates when G is controlled by the total pressure in the film is given in [4].

Introduce the rectangular coordinate system $(|x| \leqslant D,\ |y| \leqslant L)$. The boundary conditions (1.3), (1.4) acquire the form

$$z = 0 : u_x = u_y = u_z = 0; \; z = h : u_x = u_y = 0,\, u_z = dh/dt, \tag{2.1}$$

and the system (1.7) reduces to the differential equation

$$m \frac{d^2h}{dt^2} = -G + 4 \int_0^D \int_0^L (p - p_c)\, dx\, dy, \tag{2.2}$$

where m is the mass of the moving plate, and p_c is the constant pressure in the surrounding medium. In writing down (2.2) it was assumed, in addition, that the dominating contribution to the fluid stress at a boundary with a free plate is provided by the static pressure only.

Consider the approximate solution of the problem, when in (1.10) the restriction is made to the first equation only for each of the two ξ. We note that within this approximation (J = 1) the nonstationarity of the velocity and pressure fields is "manifested" only through the time dependence of the generalized coordinate h. In this case the determination of the pressure field in the region investigated reduces, with account of (1.12), (1.13), and (2.1), to solving the special case of the Reynolds equation

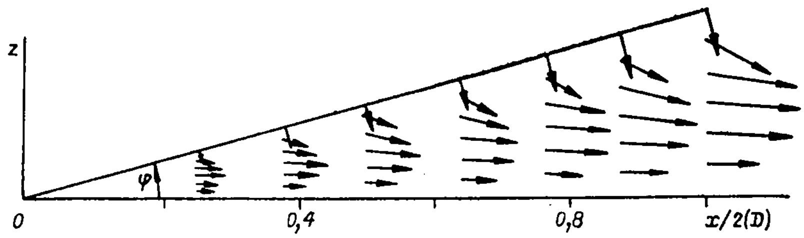

Fig. 4

$$\Delta p = \frac{12\mu}{h^3}\frac{dh}{dt} \tag{2.3}$$

(Δ is the Laplace operator). As boundary condition for the pressure we took $p = p_c$ on the external plate contour.

Solutions of (2.3) were sought in the form

$$p = p_c + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} C_{nk}\cos\frac{\varepsilon_n x}{D}\cos\frac{\varepsilon_k y}{L},\quad \varepsilon_l = \frac{\pi}{2}(2l-1)\quad (l = 1, 2, \ldots). \tag{2.4}$$

Here the expansion coefficients, taking into account the orthogonality conditions of the basis functions, are determined as follows:

$$C_{nk} = -\frac{48\mu(-1)^{n+k}}{h^3\varepsilon_n\varepsilon_k}\left(\frac{\varepsilon_n^2}{D^2}+\frac{\varepsilon_k^2}{L^2}\right)^{-1}\frac{dh}{dt}\quad n, k = 1, 2, \ldots \tag{2.5}$$

The fluid velocity distribution under the plate can be obtained within the approximation considered from (1.8)-(1.10) with account of (2.4), (2.5).

Substituting (2.4), with account of (2.5), into (2.2), following the corresponding transformations we reach an equation describing the dynamics of the free plate

$$\frac{d^2h'}{d(t')^2} = -1 - \frac{A}{(h')^3}\frac{dh'}{dt'}\quad A = \frac{192\mu D^3}{\sqrt{mGL^3}}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\varepsilon_n^2\varepsilon_k^2}\left(\varepsilon_n^2+\varepsilon_k^2\left(\frac{D}{L}\right)^2\right)^{-1}. \tag{2.6}$$

In (2.6) the differentiation of the dimensionless generalized coordinate $h' = h/L$ is carried out with respect to the dimensionless time $t' = t\sqrt{G/(Lm)}$.

As illustration we show in Fig. 1 the dependences of h' on t', obtained for $A = 0.132\cdot 10^{-3}$ by numerical solution of (2.6) with initial conditions: $t' = 0$, $h' = 0.1$, $dh'/dt' = 0$; -0.5; -1 (lines 1-3). It is seen that the process of film compression under the action of a constant force occurs conditionally in two stages. At the first stage, when the "supporting power" of the film, determined by the double integral in (2.2), is small in comparison with G, the free plate is quite quickly approximated by a fixed plate, is practically not subject to film resistance. The initial phase of this stage occurs, obviously, in a regime near that of a free plate. At the second phase, when the "supporting power" of the film becomes comparable with G, we have $d^2h'/d(t')^2 \to 0$.

We note that the study of the axially symmetric problem on fluid film compression between disks of radius R by the approach described above in a cylindrical coordinate system reduces, in the first approximation, to solution of an equation coinciding in shape to the description in (2.6). In this case the dimensionless parameters of Eq. (2.6) are defined in the form

$$h' = \frac{h}{R},\quad t' = t\sqrt{\frac{G}{Rm}},\quad A = \frac{3\pi\mu}{2}\sqrt{\frac{R^3}{mG}}.$$

3. Consider the compression process in a gravity force field of a viscous incompressible fluid of a thin rectangular homogeneous plate of mass m immersed in this fluid, which is freely based on some fixed plane, so that some tapered region is formed between the plane and the plate. We introduce a rectangular coordinate system, as shown in Fig. 2. Despite the fact that the plate is freely based on a plane, it is assumed that during the fluid compression from the inclined region there is no displacement (gliding) of the plate

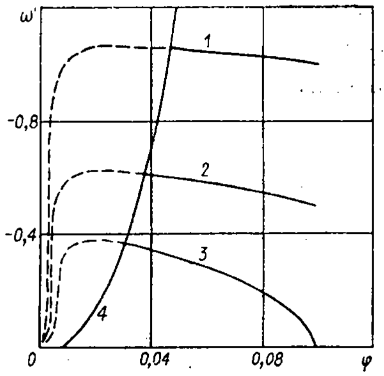

Fig. 5

relative to the plane along the x axis. It is assumed that the maximum distance between the plate and the fixed plane is quite short in comparison with the plate sizes, and the fluid flow in the inclined region can be described by the system (1.1), (1.2) with account of all assumptions made earlier. As boundary conditions for the fluid velocity along the transverse coordinate we use (1.3), (1.4), taking into account that for the problem investigated

$$h(x,\ t) = x\tan\varphi,\ V_x = V_y = V_z \equiv 0,$$
$$W_x(x,\ t) = -x\tan\varphi\frac{d\varphi}{dt},\quad W_y \equiv 0,\quad W_z(x,\ t) = x\frac{d\varphi}{dt} \tag{3.1}$$

(φ is the generalized coordinate of the plate, being the angle between the plate and the fixed plane). In addition to (1.3), (1.4), we write down the boundary and initial conditions

$$x = 2D\cos\varphi,\ p = p_c,$$
$$y = \pm L,\ p = p_c; \tag{3.2}$$
$$x = 0,\ p \neq \infty,$$
$$t = 0,\ \varphi = \varphi_0,\ d\varphi/dt = \omega_0, \tag{3.3}$$

where 2D, 2L are the plate sizes, respectively, in the directions of the x and y axes, p_c is the fluid pressure in the surrounding space of the plate, and φ_0, ω_0 are the initial values of the resolving angle of the inclined flow region and the angular velocity of the plate.

Consider the simplest special case, when in constructing the approximate solution one is confined in (1.10) to the first equation for each of the two ξ. The pressure field in the inclined region can then be found from the Reynolds equation

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right) = 12\mu W_z, \tag{3.4}$$

whose solution is sought in the form

$$p = p_c + \sum_{n=1}^{\infty} C_n\cos\frac{\varepsilon_n y}{L},\quad \varepsilon_n = \frac{\pi}{2}(2n-1) \tag{3.5}$$

(C_n are the yet unknown expansion coefficients, being functions of the coordinate x and time t).

Substituting (3.5) into (3.4), and applying the orthogonality condition of the basis functions, following transformations with account of (3.1) we reach an ordinary differential equation in C_n:

$$x^2\frac{d^2C_n}{dx^2} + 3x\frac{dC_n}{dx} - \left(\frac{\varepsilon_n x}{L}\right)^2 C_n = \frac{24\mu}{\varepsilon_n\tan^3\varphi}(-1)^{n+1}\frac{d\varphi}{dt},\quad n = 1,\ 2,\ \ldots \tag{3.6}$$

We note that the time dependence of C_n, and consequently, the velocity and pressure fields, is manifested within the approximation considered (J = 1) through the time dependence of the generalized coordinates and velocities of the plate. The general solution of (3.6) is

$$C_n = \frac{24\mu}{\varepsilon_n^3 \tan^3\varphi}\left(\frac{L}{x}\right)^2 (-1)^n \frac{d\varphi}{dt} + \frac{1}{x}\left[c_{1n} I_1\left(\frac{\varepsilon_n x}{L}\right) + c_{2n} K_1\left(\frac{\varepsilon_n x}{L}\right)\right], \quad n = 1, 2, \ldots \tag{3.7}$$

(c_{1n}, c_{2n} are unknown integration constants). Here and in the following I_l, K_l are the modified Bessel function and Macdonald function of order l.

Taking into account the boundary conditions (3.2), (3.3), the behavior of the functions I_1 and K_1 for $\varepsilon_n x/L \ll 1$ [6], as well as the a priori assumed convergence for all $|y| \leqslant L$ of the series $\sum_{n=1}^{\infty} c_{1n}\varepsilon_n \cos\frac{\varepsilon_n y}{L} \neq \infty$, it can be shown that the integration constants in (3.7) are found in the form

$$c_{1n} = \frac{24\mu L E_n (-1)^{n+1}}{\varepsilon_n^2 \tan^3\varphi}\frac{d\varphi}{dt}, \quad c_{2n} = \frac{c_{1n}}{E_n},$$
$$E_n = \left[\alpha_n^{-1} - K_1(\alpha_n)\right] I_1^{-1}(\alpha_n), \quad \alpha_n = \frac{2D\varepsilon_n \cos\varphi}{L}, \quad n = 1, 2, \ldots \tag{3.8}$$

The final determination within the approximation considered of the pressure field, and consequently, with account of (1.8)-(1.10), of the velocity field in the inclined region reduces to finding the time dependence of the generalized coordinate φ from system (1.7), which is transformed to the differential equation

$$J_y \frac{d^2\varphi}{dt^2} = -Dmg\cos\varphi + 2\int_0^L \int_0^{2D\cos\varphi} (p - p_c)\frac{x\,dx\,dy}{\cos^2\varphi}, \tag{3.9}$$

where g is the free fall acceleration, and $J_y = 4mD^2/3$ is the moment of inertia of the plate with respect to the y axis. In deriving (3.9) it is assumed in addition that the force action of the fluid on the plate reduces to the static pressure only. Substituting now expression (3.5) into (3.9) with account of (3.7), (3.8), following several transformations we reach an equation for determining $\varphi = \varphi(t)$:

$$\frac{d^2\varphi}{d(t')^2} = -\cos\varphi - \frac{A_1(\varphi)}{\tan^3\varphi}\frac{d\varphi}{dt'} \tag{3.10}$$

($t' = t\sqrt{mgD/J_y}$ is dimensionless time). For brevity of description we put in (3.10)

$$A_1(\varphi) = -\frac{B}{\cos^2\varphi}\left(\frac{L}{D}\right)^3 \sum_{n=1}^{\infty} \frac{1}{\varepsilon_n^4}\left[\ln\frac{2}{\gamma\alpha_n} - K_0(\alpha_n) + E_n(I_0(\alpha_n) - 1)\right],$$
$$B = \frac{72\mu}{m}\sqrt{\frac{D^3}{3g}} \tag{3.11}$$

($\gamma = 1.781\ldots$ is the Euler constant).

As mentioned above, since flows are considered in a region whose transverse sizes are small in comparison with its longitudinal sizes, it can be assumed that $\varphi \ll 1$, putting

$$\sin\varphi \approx \varphi, \quad \cos\varphi \approx 1. \tag{3.12}$$

Taking into account the last replacement, Eq. (3.10) is represented in the form

$$\frac{d^2\varphi}{d(t')^2} = -1 - \frac{A}{\varphi^3}\frac{d\varphi}{dt'}, \quad A = A_1(0). \tag{3.13}$$

It is noted that structurally the description (3.13) coincides with (2.6). The same can be said concerning the solutions of these equations for identical initial conditions and equal values of the parameter A. Therefore, the nature of variation in φ with the flow of dimensionless time t' can be illustrated by the dependences in Fig. 1.

Determining during the numerical solution of (3.13) the time dependence of the generalized coordinates, with account of (1.10), (1.12), (1.13), (3.5), (3.7), (3.8) we reach equations, determining the pressure and velocity fields in the inclined flow region:

$$p = p_c + \frac{24\mu}{\varphi^3}\frac{d\varphi}{dt}\sum_{n=1}^{\infty} F_{1n}\left(\frac{x}{L}\right)\cos\left(\frac{\varepsilon_n y}{L}\right),$$

$$u_x = -z\frac{d\varphi}{dt} + \frac{12(z^2 - xz\varphi)}{\varphi^3 L}\frac{d\varphi}{dt}\sum_{n=1}^{\infty} F_{2n}\left(\frac{x}{L}\right)\cos\left(\frac{\varepsilon_n y}{L}\right),$$

$$u_y = -\frac{12(z^2 - xz\varphi)}{\varphi^3 L}\frac{d\varphi}{dt}\sum_{n=1}^{\infty}\varepsilon_n F_{1n}\left(\frac{x}{L}\right)\sin\left(\frac{\varepsilon_n y}{L}\right), \qquad (3.14)$$

$$u_z = \frac{(3x\varphi z^2 - 2z^3)}{x^2\varphi^3}\frac{d\varphi}{dt} + \frac{12(z^3 - x\varphi z^2)}{x\varphi^3 L}\frac{d\varphi}{dt}\sum_{n=1}^{\infty} F_{2n}\left(\frac{x}{L}\right)\cos\left(\frac{\varepsilon_n y}{L}\right).$$

Here

$$F_{1n}\left(\frac{x}{L}\right) = \frac{(-1)^{n+1}}{\varepsilon_n^2}\left\{-\frac{1}{\varepsilon_n}\left(\frac{L}{x}\right)^2 + \left(\frac{L}{x}\right)\left[E_n I_1\left(\frac{\varepsilon_n x}{L}\right) + K_1\left(\frac{\varepsilon_n x}{L}\right)\right]\right\};$$

$$F_{2n}\left(\frac{x}{L}\right) = \frac{(-1)^{n+1}}{\varepsilon_n^2}\left\{\frac{2}{\varepsilon_n}\left(\frac{L}{x}\right)^3 - 2\left(\frac{L}{x}\right)^2 K_1\left(\frac{\varepsilon_n x}{L}\right) - \varepsilon_n\left(\frac{L}{x}\right)K_0\left(\frac{\varepsilon_n x}{L}\right) + \right.$$

$$\left. + E_n\left[\varepsilon_n\left(\frac{L}{x}\right)I_0\left(\frac{\varepsilon_n x}{L}\right) - 2\left(\frac{L}{x}\right)^2 I_1\left(\frac{\varepsilon_n x}{L}\right)\right]\right\} \quad (n = 1, 2, \ldots).$$

It follows from (3.14) that the time dependence of the excess pressure field ($p - p_c$) is determined by the cofactor $\varphi^{-3}d\varphi/dt$, which in turn is linearly related, taking into account (3.13), with the angular acceleration of the plate. As an example we show in Fig. 3 lines of equal dimensionless excess pressure $4(p - p_c)LD/(mg)$ in the inclined region, constructed with account of (3.13), (3.14) for $D/L = 1$, when the angular acceleration of the plate is $d^2\varphi/d(t')^2 = -0.9$. Figure 4 shows the dimensionless velocity field in the cross section $y = 0$ of the inclined region, constructed with account of (3.14) for $D/L = 1$ and $\varphi = 0.262$. As the velocity scale at a given moment of time we took the value $-2Dd\varphi/dt$, corresponding to the linear velocity of the plate points most removed from the axis of rotation.

Consider the problem of validity limits of the relations obtained (3.14), for which we used, with account of (3.11), the theorem of center of mass motion of the plate in the projection on the z axis:

$$mD\left(\frac{d^2\varphi}{dt^2} - \varphi\left(\frac{d\varphi}{dt}\right)^2\right) = -mg + N + F \qquad (3.15)$$

(N is the plane reaction, on which the plate is based). Taking into account (3.14), in (3.15) we took for brevity

$$F = 2\int_0^L\int_0^{2D}(p - p_c)\,dx\,dy = \frac{48\mu L^2 B_1}{\varphi^3}\frac{d\varphi}{dt}.$$

Following the transformations and estimates of [6, 7], B_1 is

$$B_1 = \sum_{n=1}^{\infty}\frac{1}{\varepsilon_n^3}\left\{\alpha_n^{-1} - K_1(\alpha_n) - \alpha_n K_0(\alpha_n) - \frac{\pi\alpha_n}{2}\left[K_0(\alpha_n)\mathbf{L}_1(\alpha_n) - K_1(\alpha_n)\mathbf{L}_0(\alpha_n)\right] + \right.$$

$$\left. + E_n\left[\alpha_n I_0(\alpha_n) - I_1(\alpha_n) + \frac{\pi\alpha_n}{2}\left[I_0(\alpha_n)\mathbf{L}_1(\alpha_n) - I_1(\alpha_n)\mathbf{L}_0(\alpha_n)\right]\right]\right\}$$

($\mathbf{L}_l$ is the modified Struve function of order l).

For a known plate law of motion $\varphi = \varphi(t)$, determined by solving (3.13), Eq. (3.15) makes it possible to find the unknown value of the plane reaction N. In this case N = 0 obviously corresponds to the moment in which the plate becomes freely based on the plane, "buoyant" on the hydrofoil. Starting with this moment of time, the plate behavior can no longer be described by the solution of Eq. (3.13), since the plate acquires additional degrees of freedom.

The plate "buoyancy" condition on the hydrofoil can be obtained for N = 0 by eliminating $d^2\varphi/dt^2$ from (3.15) by means of (3.13), and following transformations it can be represented in dimensionless form

$$(\omega')^2\varphi + B_2\varphi^{-3}\omega' - \frac{1}{3} = 0, \quad B_2 = A + \frac{4}{3}BB_1\left(\frac{L}{D}\right)^2, \quad \omega' = \frac{d\varphi}{dt'}. \qquad (3.16)$$

As an example we provide in Fig. 5 the dependences of ω' on φ, obtained during the numerical solution of (3.13), for the following initial conditions: $t' = 0$, $\varphi = 0.1$, $\omega' = -1; -0.5; 0$ (lines 1-3). The calculations were performed for $B = 10^{-4}$ for a plate with $D/L = 1$, corresponding to $A = 0.132 \cdot 10^{-4}$, $B_2 = -0.274 \cdot 10^{-4}$. For the same values of the original parameters we derived, with account of (3.16), the boundary (curve 4) of the region of and ω' values, in achieving which the plate is "buoyant" on the hydrofoil. The primed portions of curves 1-3, though formally corresponding to Eq. (3.13), cannot be realized. This is due to the fact that the plate behavior following the moments of "buoyancy" corresponding to the intersection points of lines 1-3 and 4 are no longer described by Eq. (3.13).

The results obtained in the examples considered for the simplest special case, when in (1.10) attention is restricted to the first equation only for each of the two $\xi(J = 1)$, are, naturally, of approximate nature. Nevertheless, it is possible to find preliminary estimates of flow characteristics. However, when higher accuracy is required, it is suggested to use in (1.10) a larger number of equations $(J \geqslant 2)$.

LITERATURE CITED

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PLANAR SURFACE WAVE GENERATION IN THE PRESENCE OF SLIGHT BOTTOM ROUGHNESS

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At present a linear theory of surface wave generation by various perturbations in a liquid with horizontal bottom has been developed quite well. However in the case of a liquid with rough bottom analytical studies of this problem have met with severe mathematical difficulties. The perturbation method is usually used for slight bottom roughness [1].

Using a linear formulation, the present study will investigate the effect of slight localized bottom roughness on the behavior of surface waves for two problems: decay of an initial elevation of the free surface and motion of a surface pressure region. A comparison is performed with a numerical solution of the original problem, obtained by the finite difference method.

1. Let an ideal incompressible homogeneous liquid occupy the region $-\infty < x < \infty$, $-H(x) \leqslant y \leqslant 0$, where x is the horizontal, and y, the vertical coordinate, $H(x) = H_0 - h(x)$, $h(x) \to 0$ as $|x| \to \infty$. At the initial moment $t = 0$ the free liquid surface is displaced from its equilibrium horizontal form and the expression $y = f_0(x)$ is specified. The velocity potential of the given flow $\varphi(x, y, t)$ satisfies the equation

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